CONCURSUL INTERJUDEŢEAN DE MATEMATICĂ

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BAREM

Clasa a X a

1. Rezolvaţi în mulţimea numerelor reale ecuaţia: $\sqrt{3^x\left(12^x-3^x\right)}+\sqrt{3^x\left(4^x-3^x\right)}=\sqrt{48^x}, x>0$.

Prof.Costel Cerchez, Brăila

***Soluţie*:**

Notăm $12^x=a>1, 4^x=b>1, 3^x=c>1, a>c, b>c \Rightarrow \sqrt{c(a-c)}+\sqrt{c(b-c)}=\sqrt{ab} \ \ |^2, (1p) \Rightarrow$

$ac+bc-2c^2+2c\sqrt{(a-c)(b-c)}=ab \Rightarrow 2c\sqrt{(a-c)(b-c)}=(a-c)(b-c)+c^2 \Rightarrow$

$\left(\sqrt{(a-c)(b-c)}-c\right)^2=0, (2p)$

$\Rightarrow \sqrt{(a-c)(b-c)}-c=0 \Rightarrow (a-c)(b-c)=c^2 \Rightarrow c^2=ab-bc-ac+c^2 \Rightarrow ab-bc-ac=0, (1p) \Rightarrow$

$\Rightarrow ab=bc+ac$

Deci $12^x+36^x=48^x \Rightarrow \left(\frac{1}{4}\right)^x+\left(\frac{3}{4}\right)^x=1 \Rightarrow f(x)=\left(\frac{1}{4}\right)^x+\left(\frac{3}{4}\right)^x$ strict descrescătoare $(2p)$

rezultă $x=1$ soluţie unică. $(1p)$

2. Să se arate că $\frac{a^2}{b^3(a+b)}+\frac{b^2}{c^3(c+b)}+\frac{c^2}{a^3(a+c)}\geq\frac{9}{2\left(a^2+b^2+c^2\right)}$, oricare ar fi $a,b,c>0$.

Profesori Gheorghe Alexe şi George-Florin Şerban , Brăila

***Soluţie*:**

$$\frac{a^2}{b^3(a+b)}+\frac{b^2}{c^3(c+b)}+\frac{c^2}{a^3(a+c)}=\frac{a^2}{b^2\left(ab+b^2\right)}+\frac{b^2}{c^2\left(c^2+bc\right)}+\frac{c^2}{a^2\left(a^2+ac\right)}=s, (1p)$$

$$s=\frac{\left(\frac{a}{b}\right)^2}{ab+b^2}+\frac{\left(\frac{b}{c}\right)^2}{cb+c^2}+\frac{\left(\frac{c}{a}\right)^2}{ac+a^2}\geq\frac{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2}{a^2+b^2+c^2+ab+bc+ac}\geq\frac{\left(3\sqrt[3]{\frac{a}{b}\frac{b}{c}\frac{c}{a}}\right)^2}{2\left(a^2+b^2+c^2\right)}=\frac{9}{2\left(a^2+b^2+c^2\right)}$$, am aplicat inegalitatea mediilor $(2p)$, inegalitatea lui Bergstrom $(2p)$ şi inegalitatea cunoscută

$a^2+b^2+c^2 \geq ab+bc+ac.(2p)$

3. Fie $\varepsilon = \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}, z\in\mathbb{C}, |z|=1, \text{Re}(z)>0$ şi $z_1 = x+iy, |z_1|=1, x\in\left[-1,-\frac{1}{2}\right], y\in\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$ şi $|z-\varepsilon|=|z-\varepsilon^2|$. Demonstraţi că $|z-z_1|=|z_1-\varepsilon|+|z_1-\varepsilon^2|$.

Prof.Carmen şi Viorel Botea, Brăila

Soluţie:

Din ipoteză avem $|z-\varepsilon|=|z-\varepsilon^2| \Leftrightarrow (z-\varepsilon)(\bar{z}-\bar{\varepsilon})=(z-\varepsilon^2)(\bar{z}-\overline{\varepsilon^2})(1p), \bar{\varepsilon}=\varepsilon^2(1p)$. Atunci

$z\bar{z}-z\bar{\varepsilon}-\bar{z}\varepsilon+\varepsilon^3 = z\bar{z}-z\varepsilon-\bar{z}\varepsilon^2+\varepsilon^3 \Big| :(-\varepsilon) \Rightarrow z\varepsilon+\bar{z}=z+\bar{z}\varepsilon \Leftrightarrow z\underbrace{(\varepsilon-1)}_{\neq 0}=\bar{z}(\varepsilon-1) \Rightarrow z=\bar{z} \Leftrightarrow z\in\mathbb{R}.(1p)$

Cum $\text{Re}(z)>0$ şi $|z|=1 \Rightarrow z^2=1 \Rightarrow z=1(1p)$. Fie $A(1), B(\varepsilon), C(\varepsilon^2) \Rightarrow \triangle ABC$ este echilateral (1*p*). Fie $M(z_1)$. Atunci M se află pe cercul $\mathcal{C}(O,1)$, pe arcul mic $BC(1p)$. Din teorema lui Pompei avem $MA=MB+MC \Rightarrow |z_1-1|=|z_1-\varepsilon|+|z_1-\varepsilon^2|(1p)$.